# Math 142 Lecture 10 Notes

Daniel Raban

February 15, 2018

## 1 The Fundamental Group

### 1.1 Group structure of homotopy classes

Recall that the relation $f \simeq g$ rel $A$ is an equivalence relation.

**Definition 1.1.** If $X$ is a topological space, and $p \in X$ is a point, we define the *fundamental group of $X$ based at $p$* to be the set $\pi_1(X, p)$ of homotopy classes rel $\{0, 1\}$ of continuous paths from $p$ to $p$; i.e.

$$\pi_1(X, p) = \{[\gamma] : (\gamma : [0, 1] \to X) \text{ is continuous}, \gamma(0) = \gamma(1) = p\},$$

and $[\gamma] = [\gamma'] \iff \gamma \simeq \gamma'$ rel $\{0, 1\}$.

**Proposition 1.1.** *$\pi_1(X, p)$ is a group under the group operation $[\alpha][\beta] = [\alpha \cdot \beta]$, where $\alpha \cdot \beta : [0, 1] \to X$ is*

$$x \mapsto \begin{cases} \alpha(2x) & x \in [0, 1/2] \\ \beta(2x - 1) & x \in (1/2, 1]. \end{cases}$$

*Proof.* We need to check that this operation is well-defined, i.e. if $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, then $[\alpha \cdot \beta] = [\alpha' \cdot \beta']$. So if $\alpha \simeq_F \alpha'$ rel $\{0, 1\}$ and $\beta \simeq_G \beta'$ rel $\{0, 1\}$, let

$$H(x, t) = \begin{cases} F(2x, t) & x \in [0, 1/2] \\ G(2x - 1, t) & x \in (1/2, 1]. \end{cases}$$

Then $\alpha \cdot \beta \simeq_H \alpha' \cdot \beta'$ rel $\{0, 1\}$, which is what we needed.

We need to check the group axioms:

1. Closure: From the definition, we get a path from $p$ to $p$.

2. Identity: Let $e : [0, 1] \to X$ be $e(x) = p$ for all $x \in [0, 1]$. Then $\alpha \cdot e$ is like $\alpha$ for $t$ up to $1/2$, and then it just stays at $p$. We want to "slide" the $1/2$ mark over closer to 1. So let

$$F(x, t) = \begin{cases} \alpha & x \in [0, 1/2 + t/2] \\ p & x \in (1/2 + t/2, 1]. \end{cases}$$

Then $\alpha \cdot e \simeq_f \alpha$ rel $\{0,1\}$. Similarly, $e \cdot \alpha \simeq \alpha$ rel $\{0,1\}$. So $[\alpha][e] = [\alpha] = [e][\alpha]$.

3. Inverses: Given a path $\alpha$ from $p$ to $p$, let $\alpha^{-1}[0,1] \to X$ is $x \mapsto \alpha(1-x)$; this is running the path backwards. The idea here is that $\alpha \cdot \alpha^{-1}$ goes from $p$ to $p$ along $\alpha$ and then goes backwards; we want to start going backwards at $\alpha(1-t)$ and then increase $t$. So let

$$F(x,t) = \begin{cases} \alpha((2-t)x) & x \in [0, 1/2 + t/2] \\ \alpha^{-1}((2-2t)x + (2t-1)) & x \in (1/2 + t/2, 1]. \end{cases}$$

Then $\alpha \cdot \alpha^{-1} \simeq_F e$ rel $\{0,1\}$. Similarly, $\alpha^{-1} \cdot \alpha \simeq_F e$ rel $\{0,1\}$. So $[\alpha][\alpha^{-1}] = [e] = [\alpha^{-1}][\alpha]$.

4. Associativity: The idea is that $(\alpha \cdot \beta) \cdot \gamma$ acts as $\alpha$ and then $\beta$ on the first two $1/4$ intervals and then $\gamma$ on the last $1/2$; $\alpha \cdot (\beta \cdot \gamma)$ acts as $\alpha$ for the first $1/2$ and then $\beta$ and then $\gamma$ on the later two $1/4$ intervals. Instead of defining $F$ directly, let $f : [0,1] \to [0,1]$ be

$$f(x) = \begin{cases} 2x & x \in [0, 1/4] \\ x + 1/4 & x \in (1/4 + 1/2] \\ \frac{x+1}{2} & x \in (1/2, 1]. \end{cases}$$

Then $((\alpha \cdot \beta) \cdot \gamma)(x) = (\alpha \cdot (\beta \cdot \gamma))(f(x))$. Note that since $[0,1]$ is convex, $f \simeq \mathrm{id}_{[0,1]}$ rel $\{0,1\}$ via the straight-line homotopy. So

$$\begin{aligned} (\alpha \cdot \beta) \cdot \gamma &= (\alpha \cdot (\beta \cdot \gamma)) \circ f \\ &\simeq (\alpha \cdot (\beta \cdot \gamma)) \circ \mathrm{id}_{[0,1]} \text{ rel } \{0,1\} \\ &= \alpha \cdot (\beta \cdot \gamma). \end{aligned}$$

□

## 1.2 Changing the basepoint

Does the fundamental group depend on the choice of point $p$?

**Definition 1.2.** $A \subseteq X$ is a *path component* if $A$ is path-connected and for any $B$ with $A \subsetneq B$, $B$ is not path connected.

If $[\gamma] \in \pi_1(X, p)$, then for all $x \in [0,1]$ $\gamma(x)$ is in the same path component of $X$ as $p$. A priori, $\pi_1(X,p)$ depends on $p$ and on the path component $p$ is in.

**Theorem 1.1.** *If $X$ is path-connected, then $\pi_1(X,p) \cong \pi_1(X,q)$ for all $p, q \in X$.*

*Proof.* We can compose paths $\gamma, \gamma'$ if $\gamma(1) = \gamma'(0)$. Similarly to in the previous proof,

$$(\gamma \cdot \gamma')(x) = \begin{cases} \gamma(2x) & x \in [0, 1/2] \\ \gamma'(2x-1) & x \in (1/2, 1] \end{cases}$$

specifies a well-defined and associative operation with inverses (up to homotopy). So choose a path $\gamma : [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$. Define a map $\gamma_* : \pi_1(X,p) \to \pi_1(X,q)$ taking $[\alpha] \mapsto [\gamma^{-1} \cdot \alpha \cdot \gamma]$.

We need to check that $\gamma_*$ is well-defined: this is true as composition is well-defined on homotopy classes. To check that $\gamma_*$ is a homomorphism, note that

$$\begin{aligned}
\gamma^{-1} \cdot (\alpha \cdot \beta) \cdot \gamma &\simeq (\gamma^{-1} \cdot \alpha) \cdot (\beta \cdot \gamma) \text{ rel } \{0,1\} \\
&= (\gamma^{-1} \cdot \alpha) \cdot e \cdot (\beta \cdot \gamma) \\
&\simeq (\gamma^{-1} \cdot \alpha) \cdot (\gamma \cdot \gamma^{-1}) \cdot (\beta \cdot \gamma) \text{ rel } \{0,1\} \\
&\simeq (\gamma^{-1} \cdot \alpha \cdot \gamma) \cdot (\gamma^{-1} \cdot \beta \cdot \gamma) \text{ rel } \{0,1\},
\end{aligned}$$

so $\gamma_*([\alpha][\beta]) = \gamma_*([\alpha])\gamma_*([\beta])$. The homomorphism $\gamma_*$ is an isomorphism because it has the inverse $(\gamma^{-1})_*$. □

This allows us to write $\pi_1(X)$ for a path-connected space $X$.